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An Asymptotic Theory for Retarded Functional Difference Equations

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Abstract—Using discrete dichotomies and Krasnoselsky's theorem, we obtain existence and asymptotic behavior of convergent solutions for retarded functional difference equations. We also will get some global properties for the set of convergent solutions. Applications on Volterra difference equations with infinite delay are shown. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In this work, we are concerned with the following homogeneous retarded linear functional equation:

$$x(n+1) = L(n, x_n), \qquad n \geq n_0 \geq 0, \tag{1.1}$$

and its perturbation

$$x(n+1) = L(n, x_n) + f_1(n, x_n) + f_2(n, x.), \qquad n \geq n_0 \geq 0, \tag{1.2}$$

where $L : \mathbb{N}(n_0) \times \mathcal{B} \longrightarrow \mathbb{C}^e$ is a bounded linear map with respect to the second variable, f_1 (respectively, f_2) is a $\mathbb{C}^e$-valued function defined on the product space $\mathbb{N}(n_0) \times \mathcal{B}$ (respectively, $\mathbb{N}(n_0) \times X$) under suitable condition; $\mathcal{B}$ denotes an abstract phase space that we will explain briefly later (see [1] for an outline of the general philosophy of such spaces), X is an appropriate Banach space endowed with the norm given by (2.1); $x.$ denotes the $\mathcal{B}$-valued function defined

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by $n \longrightarrow x_n$, and $\mathbb{N}(n_0)$ denotes the set $\{n \in \mathbb{N}/n \geq n_0\}$. In [2] the authors have treated the convergence problem for nonautonomous Volterra difference systems with infinite delay under a dichotomic assumption for the solution operator, which is associated with the homogeneous linear Volterra system. We generalize some results and techniques used in [2] for a much more general context by considering the general class (1.1),(1.2). We treat the existence and asymptotic behavior of convergent solutions for a retarded functional difference equation submitted to two perturbations f_1 and f_2, f_1 locally Lipschitz type and f_2 strongly nonlinear under the hypothesis that the solution operator of equation (1.1) has an ordinary dichotomy (or (k_1, k_2)-dichotomy). All our methods are applicable for linear perturbations of equation (1.1) (see Remark 3.2) at the same time we obtain interesting information about the set of convergent solutions of equation (1.2) (see Remarks 3.3 and 3.7, also Theorems 3.2 and 3.4). Among the techniques used to study this kind of problem, the topological method that we have chosen is Krasnoselsky's theorem; to apply it, we need very detailed knowledge about the relatively compact sets on the Banach space of all convergent functions from $\mathbb{N}(n_0)$ into $\mathcal{B}$ (see Section 2). So the main technical ingredient will be the compactness criterion (see Lemma 2.1). Our result turns out to be very useful to study discrete Volterra systems; we have obtained quite precise and complete results on convergent solutions (see "Applications", Section 4). The Volterra difference equations can be considered as natural generalization of difference equations. During the last few years discrete Volterra equations have emerged vigorously in several applied fields. There is much interest in developing the qualitative theory for such equations (see [3] for discussion and references); it mainly arises in the modelling of many real phenomena or by applying numerical methods for solving Volterra integral or integrodifferential equations. Moreover, Volterra systems describe processes whose current state is determined by their entire prehistory. These processes are encountered in models of propagation of perturbation in matter with memory, various problems of heredity or epidemics, theory of viscoelasticity, and to solve optimal control problems. For a very nice article on stability and boundedness of Volterra difference equations, see [3].

The convergence theory plays an important role in the investigation of many fundamental problems related to difference equations. The first results concerned with convergence solutions for ordinary difference equations were established by Cheng *et al.* [4], Drosdwicz and Popenda [5], Lakshmikantham and Trigiante [6], Aulbach [7], and Agarwal [8]. However, many of them are related to special classes of second-order difference equations. The main feature that distinguishes our approach from the classical ones is the fact that the domain of definition of the initial conditions is an abstract phase space. The abstract phase spaces were introduced by Hale and Kato [9] for studying qualitative theory of functional differential equations with unbounded delay. The idea of considering phase spaces for studying qualitative properties of functional difference equations was used first by Murakami [10] for study some spectral properties of the solution operator for linear Volterra difference systems and then by Elaydi *et al.* [11] to study asymptotic equivalence of bounded solutions of a homogenous Volterra difference system and its perturbation. From then the theory of abstract retarded functional difference equations in phase spaces has drawn the attention of several authors (see [2,10–15]).

The paper is organized as follows. The second section provides the definitions and preliminary results to be used in the theorems stated and proved in this work. In the third section, we study existence and asymptotic behavior of convergent solutions of equation (1.2) and we get some global properties for the set of convergent solutions, while in the fourth section, we present applications. Throughout this paper, we will always assume that $\mathcal{B}$ is a phase space and L is a bounded linear map with respect to the second variable.

2. PRELIMINARIES AND NOTATIONS

Here we explain notations and provides some auxiliary results that we will need in the subsequent section. The phase space $\mathcal{B} = \mathcal{B}(\mathbb{Z}^-, \mathbb{C}^e)$ is a Banach space (with norm denoted by $\|\cdot\|_{\mathcal{B}}$)

which is a subfamily of functions from $\mathbb{Z}^-$ into $\mathbb{C}^e$ and it is assumed to satisfy the following axiom.

AXIOM (A). *There are a positive constant $J > 0$ and nonnegative functions $N(\cdot)$ and $M(\cdot)$ on $\mathbb{Z}^+$ with the property that if $x : \mathbb{Z} \longrightarrow \mathbb{C}^e$ is a function, such that $x_0 \in \mathcal{B}$, then for all $n \in \mathbb{Z}^+$, the following conditions are held.*

(i) $x_n \in \mathcal{B}$.
(ii) $J|x(n)| \le \|x_n\|_{\mathcal{B}} \le N(n) \sup_{0\le s\le n} |x(s)| + M(n)\|x_0\|_{\mathcal{B}}$.

To obtain our result, we will need one additional property on $\mathcal{B}$, namely the following.

AXIOM (B). *The inclusion map $i : (B(\mathbb{Z}^-, \mathbb{C}^e), \|\cdot\|_\infty) \longrightarrow (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is continuous, i.e., there is a constant $K \ge 0$, such that $\|\varphi\|_{\mathcal{B}} \le K\|\varphi\|_\infty$, for all $\varphi \in B(\mathbb{Z}^-, \mathbb{C}^e)$ (where $B(\mathbb{Z}^-, \mathbb{C}^e)$ represents the bounded functions on $\mathbb{Z}^-$ in $\mathbb{C}^e$).*

From now on $\mathcal{B}$ will denote a phase space satisfying the Axioms (A),(B). For any $n \ge \tau$, we define the bounded linear operator $T(n,\tau) : \mathcal{B} \longrightarrow \mathcal{B}$ by $T(n,\tau)\varphi = x_n(\tau,\varphi,0)$ for $\varphi \in \mathcal{B}$, where $x(\cdot,\tau,\varphi,0)$ denotes the solution of the homogeneous linear system (1.1) passing through (τ,φ). The operator $T(n,\tau)$ is called the solution operator of the homogeneous linear system (1.1). For convenience, we will recall the definition of ordinary dichotomy (and (k_1,k_2)-dichotomy).

DEFINITION 2.1.

(a) *We say that system (1.1) has an ordinary dichotomy, if the solution operator $T(n,\tau)$ satisfies the following property: there is a positive constant $\tilde{K}$ and a projection operator $P(\tau) : \mathcal{B} \longrightarrow \mathcal{B}$, $\tau \in \mathbb{Z}^+$, such that if $Q(\tau) = I - P(\tau)$, then*
 (a_1) *$T(n,\tau)P(\tau) = P(n)T(n,\tau)$, $n \ge \tau$;*
 (a_2) *the restriction $T(n,\tau) \mid R(Q(\tau))$, $n \ge \tau$, is an isomorphism of $R(Q(\tau))$ onto $R(Q(n))\cdot$ $(R(Q(\cdot))$ denotes the range of $Q(\cdot))$ and we define $T(\tau,n)$ as the inverse mapping of $T(n,\tau)$;*
 (a_3) *$\|T(n,\tau)P(\tau)\| \le \tilde{K}$, $n \ge \tau$;*
 (a_4) *$\|T(n,\tau)Q(\tau)\| \le \tilde{K}$, $\tau \ge n$;*

(b) *Assume that Conditions (a_1) and (a_2) hold and let k_i, $i = 1,2$ be two positive sequences, we say that system (1.1) has a (k_1,k_2)-dichotomy if a positive constant M exist, such that:*
 (a_5) *$\|T(n,\tau)P(\tau)\| \le Mk_1(n)k_1(\tau)^{-1}$, $n \ge \tau$;*
 (a_6) *$\|T(n,\tau)Q(\tau)\| \le Mk_2(n)k_2(\tau)^{-1}$, $\tau \ge n$.*

(c) *We say that a (k_1,k_2)-dichotomy is compensated if, and only if, a positive constant $C \ge 1$ exists, such that $k_1(n)k_1(m)^{-1} \le Ck_2(n)k_2(m)^{-1}$, $n \ge m$.*

Denote by X the Banach space of all bounded functions $\eta : \mathbb{N}(n_0) \longrightarrow \mathcal{B}$ endowed with the norm

$$\|\eta\| = \sup_{n\ge n_0} \|\eta(n)\|_{\mathcal{B}}. \tag{2.1}$$

Also, we denote by X_∞ the Banach space all convergent functions $\xi \in X$, i.e., for which the limit $Z_\infty(\xi) := \lim_{n\to\infty} \xi(n)$ exists, endowed with the preceding norm. On the other hand, for each $\lambda > 0$ we denote by $X_\infty[\lambda]$ the ball $\|\xi\| \le \lambda$ in X_∞.

We will use the following result proved in [14].

LEMMA 2.1. COMPACTNESS CRITERION ON X_∞. *Let S be a subset of X_∞. Suppose the following conditions are satisfied.*

(C_1) *The set $H(n) := \{\xi(n)/\xi \in S\}$ is relatively compact on $\mathcal{B}$, for all $n \in \mathbb{N}(n_0)$.*
(C_2) *S is equiconvergent at ∞, i.e., for every $\varepsilon > 0$ there exists an N, such that $\|\xi(n) - Z_\infty(\xi)\|_{\mathcal{B}} < \varepsilon$ for $n \ge N$, for all $\xi \in S$.*

Then S is relatively compact on X_∞.

3. CONVERGENT SOLUTIONS

3.1. Convergent Solutions for Systems with Ordinary Dichotomy

Krasnoselsky's theorem is an useful tool in the proof of existence theorems for functional equations (see [16] for discussion and references). This powerful theorem has not been sufficiently used in difference equations. Using it, we will show the existence of convergent solutions of equation (1.2). Among other things, in the development of this section, we also will obtain interesting information about the set of convergent solutions of equation (1.2) (see Remarks 3.3 and 3.7, also Theorems 3.2 and 3.4). To make the exposition clearer, we will first discuss the ordinary dichotomy case, so in the process of obtaining our results, we initially will require that the solution operator, which is associated with the homogeneous linear equation (1.1), has an ordinary dichotomy and the quasilinear functional difference equation (1.2) being submitted to two perturbations.

Let us agree, first of all, to employ the following notations throughout the paper: $E^0(t)$, $t \in \mathbb{Z}^-$ denotes the $e \times e$ matrix defined by $E^0(t) = I$ (unit matrix) if $t = 0$; $E^0(t) = 0$ (zero matrix), if $t < 0$. $\Gamma(n, s)$ denotes the Green function associated with equation (1.1), i.e.,

$$\Gamma(n,s) = \begin{cases} T(n,s+1)P(s+1), & \text{if } n-1 \geq s, \\ -T(n,s+1)Q(s+1), & \text{if } s > n-1. \end{cases}$$

Specifically to establish the next results, we need to introduce the following assumption.

ASSUMPTION (D). *The following conditions hold.*

(d-1) *The function $f_1(n, \varphi)$ is locally Lipschitz in $\varphi \in \mathcal{B}$, i.e., for each positive number, R, for all $\varphi, \psi \in \mathcal{B}$, with $\|\varphi\|_{\mathcal{B}}, \|\psi\|_{\mathcal{B}} \leq R$, $|f_1(n,\varphi) - f_1(n,\psi)| \leq F_1(n,R)\|\varphi - \psi\|_{\mathcal{B}}$, where $F_1 : \mathbb{N}(n_0) \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing function with respect to second variable.*

(d-2) *There is a positive constant μ, such that $\sum_{s=n_0}^{\infty} F_1(s,\mu) < +\infty$.*

(d-3) *There is a positive constant λ and a function $F_2 : \mathbb{N}(n_0) \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ nondecreasing with respect to the second variable, such that: for each $(n,\xi) \in \mathbb{N}(n_0) \times X$ with $\|\xi\| \leq \lambda$, $|f_2(n,\xi)| \leq F_2(n, \|\xi\|)$.*

(d-4) *There is a positive constant $\tilde{\mu}$, such that $\beta_{\tilde{\mu}} := \sup_{\gamma \in (0,\tilde{\mu}]} \delta(\gamma)/\gamma < 1$, where $\delta(\gamma) := K\tilde{K}\sum_{s=n_0}^{\infty} F_2(s,\gamma)$, K and $\tilde{K}$ are constant of Axiom (B) and Definition 2.1(a), respectively.*

The proof of the following result combines the crucial compactness criterion from the preceding section together with Krasnoselsky's theorem (we observe that, in equation (1.2), the perturbation f_1 induces a contractive operator (see (3.1)) while that f_2 induce a compact operator (see (3.2)). We obtain existence, asymptotic behavior, and asymptotic stability (AS in short, see [1]) of convergent solution of equation (1.2), in other words, if $f_1(n,0) = f_2(n,0) = 0$ the zero solution of equation (1.2) is AS with respect to this class.

THEOREM 3.1. *Assume that Condition (D) holds. In addition, suppose the following conditions are satisfied.*

(D_1) *System (1.1) has an ordinary dichotomy, such that $\lim_{m\to\infty} T(m,n)P(n) = 0$ for every $n \in \mathbb{N}(n_0)$.*

(D_2) *For every $n \geq n_0$ the function $g(n,\cdot) := F_2(n,\tilde{\mu})^{-1} f_2(n,\cdot)$ is continuous.*

(D_3) *The limit $\pi(\xi) := Z_\infty(g(\cdot,\xi))$ exists uniformly in $\xi \in X_\infty[\lambda]$.*

Then, there are positive constants $\tilde{M}$ and $n_1 \in \mathbb{N}(n_0)$, such that for each $\varphi \in P(n_1)\mathcal{B}$ with $\|\varphi\|_{\mathcal{B}} \leq \tilde{M}$, there is a solution $y = y(\varphi) = y(n, n_1, \psi)$, with $P(n_1)\psi = \varphi$, of equation (1.2) for $n \geq n_1$, such that $y_n(\varphi) = o(1)$ as $n \to \infty$.

PROOF. For convenience we can assume that $f_1(n,0) = 0$. Using (d-2) there is $n_1 \in \mathbb{N}(n_0)$ large enough, such that: $T := \beta_{\tilde{\mu}} + K\tilde{K}(\sum_{s=n_1}^{\infty} F_1(s,\mu)) < 1$, where $\beta_{\tilde{\mu}}$ is given by (d-4). Let γ be a constant, such that $0 < \gamma < \min\{\lambda, \mu, \tilde{\mu}\}$. We consider the operator $B : X_\infty[\gamma] \longrightarrow X_\infty$ defined by

$$(B\eta)(n) := \sum_{s=n_1}^{\infty} \Gamma(n,s)E^0(f_1(s,\eta(s))). \tag{3.1}$$

A short argument involving (D_1) combined with (d-1) and (d-2) shows that $Z_\infty(B\eta) = 0$. Furthermore, for $\xi, \eta \in X_\infty[\gamma]$, we easily check that

$$\|B\eta - B\xi\| \le K\tilde{K}\left(\sum_{s=n_1}^{\infty} F_1(s,\mu)\right)\|\eta - \xi\|.$$

Let us denote by $\tilde{M} := (\gamma\beta_{\tilde{\mu}} - \delta(\gamma))\tilde{K}^{-1}$ and $\varphi \in P(n_1)\mathcal{B}$, such that $\|\varphi\|_{\mathcal{B}} \le \tilde{M}$. Next, we define an operator T on $X_\infty[\gamma]$ by

$$(T\xi)(n) := \Gamma(n, n_1 - 1)\varphi + \sum_{s=n_1}^{\infty} \Gamma(n,s)E^0(f_2(s,\xi)), \tag{3.2}$$

for all $\xi \in X_\infty[\gamma]$ and $n \ge n_0$. Take any $\xi \in X_\infty[\gamma]$, we will now prove that $T\xi \in X_\infty[\gamma]$. Indeed, we notice that $T\xi$ can be estimated as $\|T\xi\| \le \gamma\beta_{\tilde{\mu}} \le \gamma$. It is not difficult to verify that $Z_\infty(T\xi) = 0$ uniformly in $\xi \in X_\infty[\gamma]$ which leads to $T\xi \in X_\infty[\gamma]$. For all $\xi, \eta \in X_\infty[\gamma]$, we note that $T\xi + B\eta \in X_\infty[\gamma]$. In fact, it follows in view of the estimates

$$\|T\xi + B\eta\| \le \gamma\beta_{\tilde{\mu}} + (\mathcal{T} - \beta_{\tilde{\mu}})\gamma \le \gamma.$$

If we show that there exists a fixed point of $T + B$, Theorem 3.1 will be proved.

We observe that the continuity of operator T is a consequence of (D_2). In fact, to prove this statement we can proceed directly by considering a sequence $\{\xi_m\}_m$, such that $\xi_m \to \xi$ in $X_\infty[\gamma]$. We select $n_2 \ge n_0$, which is large enough, using (d-3), one arrives at the following estimate:

$$\|T\xi_m - T\xi\| \le \delta(\tilde{\mu}) \max_{n_1 \le s \le n_2 - 1} |g(s,\xi_m) - g(s,\xi)| + 2K\tilde{K}\sum_{s=n_2}^{\infty} F_2(s,\tilde{\mu}),$$

where $\delta(\tilde{\mu})$ is given by (d-4). Thus, our assertion is proved. An essential step is to show that the image of T is relatively compact in X_∞. We will first prove that $H(n) = \{(T\xi)(n) : \xi \in X_\infty[\gamma]\}$ is relatively compact in $\mathcal{B}$ for all $n \ge n_0$. We consider an arbitrary sequence $(\xi_m)_m$ in $X_\infty[\gamma]$, we get that $(g(\cdot,\xi_m))_m$ is relatively compact in l^∞ (here l^∞ denotes the Banach space of all bounded sequences from $\mathbb{N}(n_0)$ into $\mathbb{C}^e$). This can be done as follows: with the aid of (d-3) it is easy to see that this sequence is bounded. On the other hand, (D_3) guarantees that it is equiconvergent. Hence, there is a subsequence $(g(\cdot,\xi_{m_i}))_i$ uniformly convergent to someone $\psi \in l^\infty$. Putting $\varphi(n) := F_2(n,\tilde{\mu})\psi(n)$, we can verify that the sequence $(T\xi_{m_i})(n)$ converges to $\Gamma(n, n_1 - 1)\varphi + \sum_{s=n_1}^{\infty}\Gamma(n,s)E^0(\varphi(s))$. Indeed, we may show the following estimate:

$$\left\|\sum_{s=n_1}^{\infty}\Gamma(n,s)E^0\left(f_2\left(s,\xi_{m_i}\right) - \varphi(s)\right)\right\|_{\mathcal{B}} \le \delta\left(\tilde{\mu}\right)\|g(\cdot,\xi_{m_i}) - \psi\|_\infty.$$

The equiconvergence at ∞ of the image of T is an immediate consequence from the fact that the limit $Z_\infty(T\xi)$ is zero uniformly in $\xi \in X_\infty[\gamma]$. Finally, compactness criterion on X_∞ leads us to conclude that the image of T is relatively compact in X_∞. Now using Krasnoselsky's theorem, we deduce that $T + B$ has a fixed point $\xi \in X_\infty[\gamma]$, which finishes the proof. ∎

Before proceeding with the next result (see Theorem 3.2), we will concentrate in the following three remarks.

REMARK 3.1. Set $Z_\varphi(n) := T(n,n_1)P(n_1)\varphi$, $n \geq n_1 \geq n_0$. In Theorem 3.1 of [12] (respectively, Theorem 4.1 of [2]) was proved the continuity of application $\varphi \longrightarrow y_\cdot(\varphi)$ and the bicontinuity of the correspondence $y_\cdot(\varphi) \longrightarrow Z_\varphi$ for autonomous (respectively, nonautonomous) Volterra difference systems, with Lipschitz perturbation, under the assumption that the solution operator, which is associated with the homogeneous autonomous (respectively, nonautonomous) Volterra difference system, has a summable dichotomy (respectively, (k_1,k_2)-dichotomy). We should remark that in our case this result does not work anymore, except for the application $y_\cdot(\varphi) \longrightarrow Z_\varphi$ which is continuous just with conditions of Theorem 3.1. However, we can obtain continuity of previous applications if we replace (D_2) of Theorem 3.1 for the following suitable condition.

(D_4) There are a positive constant σ and a function $G : \mathbb{N}(n_0)\times\mathbb{R}^+\times\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing function with respect to the second and third variables, such that $\sum_{s=n_0}^{\infty} F_2(s,\tilde{\mu})G(s,\sigma,\sigma) < +\infty$ and $|g(n,\xi) - g(n,\eta)| \leq G(n, \|\xi\|, \|\eta\|)\|\xi - \eta\|$, for all $\xi, \eta \in X$.

For convenience of the reader we will include the proof of this fact. We first note that we can assume that $f_1(n,0) = 0$ and γ the constant in the proof of Theorem 3.1, such that $0 < \gamma < \min\{\lambda, \mu, \tilde{\mu}, \sigma\}$. Using (d-2) and (D_4) there is $n_1 \in \mathbb{N}(n_0)$ large enough, such that

$$\mathcal{T} := \beta_{\tilde{\mu}} + K\tilde{K}\left[\sum_{s=n_1}^{\infty} F_1(s,\mu) + \sum_{s=n_1}^{\infty} F_2(s,\tilde{\mu})\, G(s,\sigma,\sigma)\right] < 1.$$

Second, we have the following estimates responsible for the continuity of preceding applications:

$$\left\|\sum_{s=n_1}^{\infty} \Gamma(n,s)E^0(f_2(s,y_\cdot(\varphi)) - f_2(s,y_\cdot(\varphi_0)))\right\|_{\mathcal{B}} \leq \left(K\tilde{K}\sum_{s=n_1}^{\infty} F_2(s,\tilde{\mu})\, G(s,\sigma,\sigma)\right) \sup_{m\geq n_1} \|y_m(\varphi) - y_m(\varphi_0)\|_{\mathcal{B}}$$

and

$$\left\|\sum_{s=n_1}^{\infty} \Gamma(n,s)E^0(f_1(s,y_s(\varphi)) - f_1(s,y_s(\varphi_0)))|\right\|_{\mathcal{B}} \leq \left(K\tilde{K}\sum_{s=n_1}^{\infty} F_1(s,\mu)\right) \sup_{m\geq n_1} \|y_m(\varphi) - y_m(\varphi_0)\|_{\mathcal{B}}.$$

From the preceding estimates, one can easily infer that

$$\|y_\cdot(\varphi) - y_\cdot(\varphi_0)\| \leq \left(\frac{\tilde{K}}{(1 - \mathcal{T} + \beta_{\tilde{\mu}})}\right) \|\varphi - \varphi_0\|_{\mathcal{B}}.$$

The same kind of previous argument shows the following estimates:

$$(1 - \mathcal{T} + \beta_{\tilde{\mu}})\, \|y_\cdot(\varphi) - y_\cdot(\varphi_0)\| \leq \|Z_\varphi - Z_{\varphi_0}\| \leq (1 + \mathcal{T} - \beta_{\tilde{\mu}})\, \|y_\cdot(\varphi) - y_\cdot(\varphi_0)\|.$$

Also, we notice that $T + B$ is a $\mathcal{T}$-contraction. Alternately, if for every $n \geq n_0$ the function $\varphi \longrightarrow g(n, y_\cdot(\varphi))$ is continuous then the function $\varphi \longrightarrow y_\cdot(\varphi)$ is continuous. This last condition however does not guarantee the continuity of $Z_\varphi \longrightarrow y_\cdot(\varphi)$. This finished the discussion of remark.

REMARK 3.2. We wish to emphasize here that the preceding theorem is applicable for linear perturbations of system (1.1), i.e., $x(n+1) = L(n,x_n) + L_1(n,x_n)$, where L_1 is a bounded linear operator with respect to the second variable under some additional but natural assumption.

The following remark gives us important information about the set of convergent solutions of (1.2).

REMARK 3.3. Let us denote by $P(n_1)\mathcal{B}[\tilde{M}]$ the ball $\|\varphi\|_{\mathcal{B}} \leq \tilde{M}$ in $P(n_1)\mathcal{B}$. Under the conditions of Theorem 3.1, we can easily infer that the set Ω of all convergent solutions $y.(\varphi)$ of equation (1.2) with $\varphi \in P(n_1)\mathcal{B}[\tilde{M}]$ is equiconvergent at ∞ on X_∞. We cannot guarantee that Ω is a relatively compact set on X_∞. Nevertheless if we modifying slightly Ω let say by $\tilde{\Omega}$ as the set of all $y.(\varphi) - Z_\varphi$ with $\varphi \in P(n_1)\mathcal{B}[\tilde{M}]$ and at the same time we introducing a similar condition to (D_3) of Theorem 3.1 given by the following.

(D_5) The limit $\tilde{\pi}(\xi) := Z_\infty(\tilde{g}(\cdot, \xi))$ exists uniformly in $\xi \in X_\infty[\tilde{\lambda}]$, where $\tilde{g}(n, \xi) := F_1(n, \mu)^{-1} \cdot f_1(n, \xi(n))$.

Then, using the compactness criterion it is not hard to prove that $\tilde{\Omega}$ is relatively compact on X_∞ (the proof of this statement make use a similar construction to used in the proof of Theorem 3.1). It seems to us that until now this kind of information has not been analyzed in the existing literature about functional difference equations.

3.2. About a Generalization of Theorem 3.1

A natural question for discussion, being of an intrinsic interest, occurs when we want to relax the condition $\lim_{m\to\infty} T(m, n)P(n) = 0$, which was responsible in Theorem 3.1 for the existence of the limit of the operators B and T in (3.1) and (3.2), respectively, we can change it by $\lim_{m\to\infty} T(m, n)P(n) = L(n) \in \mathcal{L}(\mathcal{B})$.

THEOREM 3.2. *Under the conditions of Theorem 3.1 except (D_1) which is replaced by the follwing.*

(D_6) *System (1.1) has an ordinary dichotomy, such that $\lim_{m\to\infty} T(m, n)P(n) = L(n)$, for all $n \geq n_0$.*

Then, there are positive constant γ and $n_1 \in \mathbb{N}(n_0)$, such that for each $\varphi \in P(n_1)\mathcal{B}$ with $\|\varphi\|_{\mathcal{B}} \leq (\gamma\beta_{\tilde{\mu}} - \delta(\gamma))\tilde{K}^{-1}$, there is a solution $y = y(\varphi) = y(n, n_1, \psi)$, with $P(n_1)\psi = \varphi$, of equation (1.2) for $n \geq n_1$, such that the limit of $y.$ exists and $\|y.\| \leq \gamma$. The limit $y.$ is given by

$$Z_\infty(y.(\varphi)) = L(n_1)\varphi + Z_\infty\left(\sum_{s=n_1}^{\cdot-1} \Gamma(\cdot, s)E^0(f_1(s, y_s(\varphi)) + f_2(s, y.(\varphi)))\right).$$

If in addition $\sup_{n\geq n_0} \|L(n)\| < +\infty$, then

$$Z_\infty(y.(\varphi)) = L(n_1)\varphi + \sum_{s=n_1}^{\infty} L(s+1)E^0(f_1(s, y_s(\varphi)) + f_2(s, y.(\varphi)))).$$

Moreover, the set Ω of all convergent solutions of equation (1.2) with initial condition in $P(n_1) \cdot \mathcal{B}[R]$, where $R := (\gamma\beta_{\tilde{\mu}} - \delta(\gamma))\tilde{K}^{-1}$ is equiconvergent at ∞ and if (D_5) holds, the set $\tilde{\Omega}$ of all $y.(\varphi) - Z_\varphi$ with $\varphi \in P(n_1)\mathcal{B}[R]$ is relatively compact (see Remark 3.3).

PROOF. Using the notations of previous theorem, we will briefly recall some argument of the proof. For convenience we can assume that $f_1(n, 0) = 0$. Initially, we note that the limit $Z_\infty(A(\cdot, \xi))$ exists uniformly in $\xi \in X_\infty[\gamma]$, where γ is small enough and $A(m, \xi) := \sum_{s=n_1}^{m-1} \Gamma \cdot (m, s)E^0(f_2(s, \xi))$. This assertion is an immediate consequence from the following two estimates. Let n_2 be a natural number large enough: for each m and n satisfying $m \geq n \geq n_2$, we get

$$\begin{aligned} &\left\|\sum_{s=n_1}^{n_2-1} (\Gamma(n, s) - \Gamma(m, s))E^0(f_2(s, \xi))\right\|_{\mathcal{B}} \\ &\qquad \leq c(n_2)\left[\max_{n_1\leq s\leq n_2-1} \|\Gamma(n, s) - L(s+1)\| + \max_{n_1\leq s\leq n_2-1} \|\Gamma(m, s) - L(s+1)\|\right], \end{aligned}$$

where $c(n_2)$ is some constant dependent of n_2. And,

$$\left\|\sum_{s=n_2}^{n-1}(\Gamma(n,s)-\Gamma(m,s))E^0(f_2(s,\xi))\right\|_{\mathcal{B}}+\left\|\sum_{s=n}^{m-1}\Gamma(m,s)E^0(f_2(s,\xi))\right\|_{\mathcal{B}}\leq 3K\tilde{K}\sum_{s=n_1}^{\infty}F_2(s,\tilde{\mu}).$$

Thus, our assertion is proved. On the other hand, the limit $Z_\infty(B(\cdot,\eta))$ exists for all $\eta\in X_\infty[\gamma]$, where $B(n,\eta):=\sum_{s=n_1}^{n-1}\Gamma(n,s)E^0(f_1(s,\eta(s)))$. It passage can be obtained in quite an analogous way to the preceding argument. Hence, the limit of $B\eta$ is explicitly computable as $Z_\infty(B\eta)=Z_\infty(B(\cdot,\eta))$. The equiconvergence at ∞ of the image of T is an immediate consequence from the fact that $Z_\infty(T\xi)=L(n_1)\varphi+Z_\infty(A(\cdot,\xi))$ uniformly in $\xi\in X_\infty[\gamma]$.

Assume that $b:=\sup_{n\geq n_0}\|L(n)\|<+\infty$, let us denote by $A(s):=f_1(s,y_s)+f_2(s,y_\cdot)$. We will show that

$$\lim_{n\to\infty}\sum_{s=n_1}^{n-1}\Gamma(n,s)E^0(A(s))=\sum_{s=n_1}^{\infty}L(s+1)E^0(A(s)).$$

The last series is well defined since it is majorized by the convergent series $\sum_{s=n_1}^{\infty}F_1(s,\mu)$ and $\delta(\gamma)$ (see (d-4)). The proof of the preceding assertion is immediate from the following estimate: choose n_2 sufficiently large and $n\geq n_2$, we get

$$\begin{aligned}&\left\|\sum_{s=n_1}^{n-1}\Gamma(n,s)E^0(A(s))-\sum_{s=n_1}^{\infty}L(s+1)E^0(A(s))\right\|_{\mathcal{B}}\\&\qquad\leq c(n_2)\max_{n_1\leq s\leq n_2-1}\|\Gamma(n,s)-L(s+1)\|+d\sum_{s=n_2}^{\infty}(F_1(s,\gamma)+F_2(s,\gamma)),\end{aligned}$$

where $c(n_2)$ is a constant dependent of n_2 and d is a constant independent of term n. ∎

3.3. Convergent Solutions for Systems with (k_1,k_2)-Dichotomy

Until now we had analyzed the convergence problem exclusively from the ordinary dichotomies point of view. In the rest of this section, our aim will be to concentrate to investigate this problem by considering (k_1,k_2)-dichotomy. In [2] the authors have treated the convergence problem for a class of nonautonomous Volterra difference systems with infinite delay under the assumption that the solution operator has a (k_1,k_2)-dichotomy. They have used the contraction principle. However, those results are not sharp enough to include more general perturbations. Beyond doubt, investigation in that direction is technically more complicated, because it is necessary to apply other kinds of fixed-point arguments as well as using effective compactness criterions, which provide us with a sensible tool to pursue new results in such a direction.

Next, we will use the following notations: $\{k(n)\}_{n\in\mathbb{Z}^+}$ is an arbitrary positive sequence. Denote by X_k the Banach space of all k-bounded functions η from $\mathbb{N}(n_0)$ into $\mathcal{B}$ with the natural norm, i.e., $\|\eta\|_k=\|\eta k^{-1}\|$. Also, we denote by $X_{\infty,k}$ the Banach space of all k-convergent functions $\xi\in X_k$, i.e., for which the limit $Z_\infty^k(\xi):=Z_\infty(\xi k)$ exists, endowed with the norm $\|\cdot\|_k$. We have a similar compactness criterion as Lemma 2.3 for the space $X_{\infty,k}$ (see [14]). On the other hand, for each $\lambda>0$ we denote by $X_{\infty,k}[\lambda]$ the ball $\|\xi\|_k\leq\lambda$ in $X_{\infty,k}$.

To establish our results (see Theorems 3.3 and 3.4), we need to introduce the following assumption.

ASSUMPTION (D)*. *The following conditions hold.*

(d-1)* *The function $f_1(n,\varphi)$ is locally Lipschitz in $\varphi\in\mathcal{B}$, i.e., for each positive number R, for all $\varphi,\psi\in\mathcal{B}$ with $\|\varphi\|_{\mathcal{B}},\|\psi\|_{\mathcal{B}}\leq R$, $|f_1(n,\varphi)-f_1(n,\psi)|\leq k_1(n)^{-1}F_1(n,R)\|\varphi-\psi\|_{\mathcal{B}}$, where $F_1:\mathbb{N}(n_0)\times\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a nondecreasing function with respect to the second variable.*

(d-2)* *There are positive constants μ_{1j}, $j=1,2$, such that $\sum_{s=n_0}^{\infty}F_1(s,\mu_{1j}k_j(s))k_1(s+1)^{-1}<+\infty$.*

(d-3)* *There are positive constants* λ_j, $j = 1,2$ *and a function* $F_{2j} : \mathbb{N}(n_0) \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *nondecreasing with respect to the second variable, such that: for each* $(n,\xi) \in \mathbb{N}(n_0) \times X_{k_j}$ *with* $\|\xi\|_{k_j} \le \lambda_j$, $|f_2(n,\xi)| \le F_{2j}(n, \|\xi\|_{k_j})$.

(d-4)* *There are positive constants* μ_j, $j = 1,2$, *such that* $\beta_{\mu_j} = \sup_{\gamma \in (0,\mu_j]} \delta_j(\gamma)/\gamma < 1$, *where* $\delta_j(\gamma) := \Gamma_1 \sum_{s=n_0}^{\infty} F_{2j}(s,\gamma) k_1(s+1)^{-1}$, $\Gamma_1 := KMC \max\{1, Ck_2(n_0)^{-1} k_1(n_0)\}$, *K is the constant of Axiom (B) and M, C are the constants of Definition 2.1(b),(c).*

The following theorem is the counterpart of Theorem 3.1 for (k_1,k_2)-dichotomies.

THEOREM 3.3. *Assume that Condition (D)* holds. In addition, suppose the following conditions are satisfied.*

(D_1)* *System (1.1) has a* (k_1,k_2)*-dichotomy which is compensated, such that* $\lim_{m\to\infty} k_1(m)^{-1} \cdot T(m,n)P(n) = 0$, *for every* $n \in \mathbb{N}(n_0)$.

(D_2)* *For every* $n \ge n_0$ *and* $j = 1,2$ *the function* $g_j(n,\cdot) := F_{2j}(n,\mu_j)^{-1} f_2(n,\cdot)$ *is continuous.*

(D_3)* *The limit* $\pi(\xi) := Z^1_\infty(g_j(\cdot,\xi))$, $j = 1,2$ *exists uniformly in* $\xi \in X_{\infty,k_j}[\lambda_j]$.

Then, there are positive constants $\tilde{M}_j$, $j = 1,2$ *and* $n_1 \in \mathbb{N}(n_0)$, *such that for each* $\varphi \in P(n_1)\mathcal{B}$ *with* $\|\varphi\|_{\mathcal{B}} \le \tilde{M}_j$, *there exists a solution* $y^j = y^j(\varphi) = y^j(n,n_1,\psi)$ $(j = 1,2)$, *with* $P(n_1)\psi = \varphi$, *of equation (1.2) for* $n \ge n_1$, *such that* $y^j_n(\varphi) = o(k_j(n))$, *as* $n \longrightarrow \infty$.

REMARK 3.4. The conclusion of Remark 3.2 is valid for the previous result.

REMARK 3.5. The previous theorem generalizes and improves Theorem 4.1 of [2] for a very general context under less restrictive assumptions, because we have eliminated a condition strongly used in the argument of its proof, namely Condition (iv) of Theorem 4.1 [2].

REMARK 3.6. Under the conditions of the preceding result, we can see that $y^j_\cdot(\varphi) \longrightarrow Z_\varphi$ is a continuous application. Next, replace (D_2)* of Theorem 3.3 for the following condition.

(D_4)* There are positive constants μ_{2j}, $j = 1,2$ and functions $G_j : \mathbb{N}(n_0) \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ continuous and nondecreasing with respect to the second and third variables, such that $\sum_{s=n_0}^{\infty} F_{2j}(s,\mu_j) G_j(s,\mu_{2j},\mu_{2j}) k_1(s+1)^{-1} < +\infty$ and $|g_j(n,\xi) - g_j(n,\eta)| \le G_j(n, \|\xi\|_{k_j}, \|\eta\|_{k_j}) \|\xi - \eta\|_{k_j}$, for all $\xi,\eta \in X_{k_j}$.

From (D_4)*, we get the continuity of application $\varphi \longrightarrow y^j_\cdot(\varphi)$ and the bicontinuity of the correspondence $y^j_\cdot(\varphi) \longrightarrow Z_\varphi$.

In concluding this section, we will remark that a similar result to that given in Remark 3.3 can be obtained for (k_1,k_2)-dichotomies.

REMARK 3.7. Under the conditions of Theorem 3.3, we have that the set Ω of all convergent solutions $y^j_\cdot(\varphi)$ of equation (1.2) with $\varphi \in P(n_1)\mathcal{B}[\tilde{M}_j]$ is weighted equiconvergent at ∞ on X_{∞,k_j}. In addition suppose the following condition is satisfied.

(D_5)* The limits $\tilde{\pi}(\xi) := Z^1_\infty(\tilde{g}_j(\cdot,\xi))$, $j = 1,2$ exist uniformly in $\xi \in X_{\infty,k_j}[\lambda_j]$, where $\tilde{g}_j(n,\xi) := k_1(n) k_j(n)^{-1} F_1(n, \mu_{1j} k_j(n))^{-1} f_1(n,\xi(n))$.

Then, the set $\tilde{\Omega}$ of all $y^j_\cdot(\varphi) - Z_\varphi$ with $\varphi \in P(n_1)\mathcal{B}[\tilde{M}_j]$ is relatively compact on X_{∞,k_j}.

3.4. A Generalization of Theorem 3.3

We want to change the condition $T(m,n)P(n) = o(k_1(m))$ as $m \longrightarrow \infty$ in the last theorem by the condition $\lim_{m\to\infty} k_j(m)^{-1} T(m,n)P(n) = L^j(n) \in \mathcal{L}(\mathcal{B})$, $j = 1,2$. We note that if k_1 and k_2 satisfy the compensation property, then there is a positive constant $\tilde{c}$, such that $0 \le \omega := \inf_n k_1(n)/k_2(n) \le \tilde{c}$. If $\omega \ne 0$, we say that k_1 and k_2 are equivalent, in this case a straightforward computation enables us to conclude that $L^2 = \omega L^1$. Otherwise, when $\omega = 0$ (i.e., k_1 and k_2 are not equivalent) we get $L^2 = 0$. Let us agree to denote by $\omega^0 = 1$ for each nonnegative real number ω and $A_j(s) := f_1(s, y^j(\varphi)) + f_2(s, y^j_\cdot(\varphi))$.

THEOREM 3.4. *Under the conditions of Theorem 3.3 except $(D_1)^*$ which is replaced by the following.*

$(D_6)^*$ *System (1.1) has a (k_1, k_2)-dichotomy which is compensated, such that $\lim_{m\to\infty} k_j(m)^{-1} \cdot T(m,n)P(n) = \omega^{j-1}L^1(n)$, $j = 1, 2$ for all $n \geq n_0$.*

Then, there are constant γ_j, $\tilde{M}_j > 0$, $j = 1, 2$, such that for each $\varphi \in P(n_1)\mathcal{B}$ with $\|\varphi\|_{\mathcal{B}} \leq (\gamma_j\beta_{\mu_j} - \delta_j(\gamma_j))\tilde{M}_j^{-1}$, there is a solution $y^j = y^j(\varphi) = y^j(n, n_1, \psi)$ $(j = 1, 2)$, with $P(n_1)\psi = \varphi$, of equation (1.2) for $n \geq n_1$, such that the limit of $y^j_\cdot$ exists and $\|y^j_\cdot\|_{k_j} \leq \gamma_j$. The k_j-limit of $y^j_\cdot$ is given by

$$Z_\infty^{k_j}\left(y^j_\cdot(\varphi)\right) = \omega^{j-1}L^1(n_1)\varphi + Z_\infty^{k_j}\left(\sum_{s=n_1}^{\cdot-1} \Gamma(\cdot, s)E^0(A_j(s))\right).$$

On the other hand, if $b_j := \sup_{n\geq n_0} \|L^1(n)\|k_j(n) < +\infty$, then

$$Z_\infty^{k_j}\left(y^j_\cdot(\varphi)\right) = \omega^{j-1}L^1(n_1)\varphi + \omega^{j-1}\sum_{s=n_1}^{\infty} L^1(s+1)E^0(A_j(s)).$$

Moreover, the set Ω of all convergent solutions $y^j_\cdot(\varphi)$ with initial condition φ, such that $\|\varphi\|_{\mathcal{B}} \leq R_j$, where $R_j := (\gamma_j\beta_{\mu_j} - \delta_j(\gamma_j))\tilde{M}_j^{-1}$ is weighted equiconvergent at ∞ on X_{∞,k_j} and if $(D_5)^*$ holds, the set $\tilde{\Omega}$ of all $y^j_\cdot(\varphi) - Z_\varphi$ with $\|\varphi\|_{\mathcal{B}} \leq R_j$ is relatively compact (see Remark 3.7).

3.5. About the Proofs of Theorems 3.3 and 3.4

It is worth noting that the proofs of Theorems 3.3 and 3.4 follow straightforward from the following observation: if system (1.1) has a (k_1, k_2)-dichotomy which is compensated, then making the following change of variables, $x_n = k_1(n)y_n$, we can infer that the system $y(n+1) = \tilde{L}(n, y_n)$ has an ordinary dichotomy, where $\tilde{L}(n, \cdot) := (k_1(n)/k_1(n+1))L(n, \cdot)$, $T_1(n, \tau) := (k_1(\tau)/k_1(n))T(n, \tau)$, $n \geq \tau$. Also, we note that equation (1.2) is transformed in

$$y(n+1) = \tilde{L}(n, y_n) + \tilde{f}_1(n, y_n) + \tilde{f}_2(n, y_\cdot),$$

where $\tilde{f}_1(n, \varphi) := k_1(n+1)^{-1}f_1(n, k_1(n)\varphi)$, $\tilde{f}_2(n, \xi) := k_1(n+1)^{-1}f_2(n, k_1\xi)$.

The innovation in this section is that we were able to reduce a system having a (k_1, k_2)-dichotomy which is compensated in a system with ordinary dichotomy. A similar reduction can be done for the continuous case. The straightforward change in the details may safely be left to the reader. This important simplification enabled us to focus our studies on systems with ordinary dichotomies or summable dichotomies. Many questions in connection with these kinds of systems remain unanswered.

4. APPLICATIONS

We complete this work by applying our previous results to the Volterra difference systems with infinite delay. Let $A(n)$, $K(s)$, $B(n)$, $D(n, m)$, and $G(m)$ be $e \times e$ matrices defined for $n \in \mathbb{N}(n_0)$, $s \in \mathbb{Z}^+$, $m \in \mathbb{Z}^-$, and $\alpha : \mathbb{Z}^+ \longrightarrow \mathbb{R}^+$ be an arbitrary positive increasing sequence, such that

$$\sum_{n=0}^{\infty} \left[|G(-n)| + |K(n)|\right] \alpha(n) < +\infty. \tag{4.1}$$

Next, we consider the following Volterra difference system with infinite delay:

$$x(n+1) = \sum_{s=-\infty}^{n} A(n)K(n-s)x(s), \qquad n \geq n_0 \geq 0, \tag{4.2}$$

and its perturbed system

$$y(n+1) = \sum_{s=-\infty}^{n} [A(n)K(n-s) + B(n)G(s-n)|y(0)| + \nu D(n,s)|y(s)|]y(s). \tag{4.3}$$

Equations (4.2) and (4.3) are viewed as functional difference equations on the phase space $\mathcal{B}_\alpha$, where $\mathcal{B}_\alpha$ is defined as $\mathcal{B}_\alpha = \{\varphi : \mathbb{Z}^- \longrightarrow \mathbb{C}^e : \|\varphi\|_{\mathcal{B}_\alpha} := \sup_{n\in\mathbb{Z}^+}[|\varphi(-n)|/\alpha(n)] < +\infty\}$. Indeed, system (4.3) can be written as a functional difference equation in the form equation (1.2). Let us consider $\xi : \mathbb{N}(n_0) \longrightarrow \mathcal{B}_\alpha$ and $\varphi \in \mathcal{B}_\alpha$, we note that

$$L(n,\varphi) := \sum_{j=0}^{\infty} A(n)K(j)\varphi(-j),$$

$$f_1(n,\varphi) := B(n)|\varphi(-n)| \sum_{s=-\infty}^{0} G(s)\varphi(s),$$

$$f_2(n,\xi) := \sum_{\tau=-\infty}^{n_0-1} \nu D(n,\tau)\,|[\xi(n_0)](\tau-n_0)|\,[\xi(n_0)](\tau-n_0) + \sum_{\tau=n_0}^{n} \nu D(n,\tau)\,|[\xi(\tau)](0)|\,[\xi(\tau)](0).$$

In order to pass to the next result let us introduce the following notations:

$$\tilde{\alpha}(\tau) := \begin{cases} \alpha(0), & \text{if } \tau \ge n_0, \\ \alpha(n_0-\tau), & \text{if } \tau < n_0. \end{cases}$$

$$\ell(n) := \sum_{\tau=-\infty}^{n} |nD(n,\tau)|\,(\tilde{\alpha}(\tau))^2.$$

THEOREM 4.1. *Suppose the following conditions are satisfied.*

(E_1) *System (4.2) has an ordinary dichotomy, such that* $\lim_{m\to\infty} T(m,n)P(n) = 0$, *for every* $n \in \mathbb{N}(n_0)$.

(E_2) $K\tilde{K}|\nu|\|\ell\|_1 < 1$, *where* ν, K, *and* $\tilde{K}$ *are the constant of (4.3), Axiom (B), and Definition 2.1(a), respectively.*

(E_3) $\sum_{s=n_0}^{\infty} \alpha(s)|sB(s)| < +\infty$.

Then, there are positive constants $\tilde{M}$ *and* $n_1 \in \mathbb{N}(n_0)$, *such that for each* $\varphi \in P(n_1)\mathcal{B}$ *with* $\|\varphi\|_{\mathcal{B}} \le \tilde{M}$, *there is a solution* $y = y(\varphi) = y(n, n_1, \psi)$, *with* $P(n_1)\psi = \varphi$, *of equation (4.3) for* $n \ge n_1$, *such that* $y_n(\varphi) = o(1)$ *as* $n \longrightarrow \infty$. *On the other hand, the correspondence* $y_\cdot(\varphi) \leftrightarrow Z_\varphi$ *is bicontinuous and the application* $\varphi \longrightarrow y_\cdot(\varphi)$ *is continuous (see Remark 3.1). Moreover, the set* Ω *of all convergent solutions* $y_\cdot(\varphi)$ *of equation (4.3) with* $\varphi \in P(n_1)\mathcal{B}[\tilde{M}]$ *is equiconvergent at* ∞ *on* X_∞ *and the set* $\tilde{\Omega}$ *defined in Remark 3.3 is relatively compact on* X_∞.

PROOF. Let us denote by $\rho := \sum_{j=0}^{\infty} |G(-j)|\alpha(j)$. A straightforward computation shows that

$$|f_1(n,\varphi) - f_1(n,\psi)| \le |B(n)|\alpha(n)\rho\,(\|\varphi\|_{\mathcal{B}_\alpha} + \|\psi\|_{\mathcal{B}_\alpha})\,\|\varphi-\psi\|_{\mathcal{B}_\alpha}$$

and

$$|f_2(n,\xi)| \le |\nu|\ell(n)\|\xi\|^2, \qquad \text{for } n \ge n_0, \quad \xi \in X.$$

Hence, Assumption (D) of Theorem 3.1 is held. We obtain (D_2) as an immediate consequence of estimate

$$|f_2(n,\xi) - f_2(n,\eta)| \le 2(1+\|\eta\|)|\nu|\left(\frac{\ell(n)}{n}\right)\|\xi-\eta\|, \tag{4.4}$$

for all $\xi, \eta \in X$ with $\|\xi - \eta\| < 1$.

Thus, all conditions of Theorem 3.1 are fulfilled. Now, using Remarks 3.1 and 3.3, we conclude the proof of Theorem 4.1. ∎

REMARK 4.1. We can get without difficulty a similar result for linear perturbations of system (4.2) (see Remark 3.2).

Our next result can be considered as a generalization of the preceding theorem (the proof of many of the steps imitates almost exactly the proof of the previous theorem). We consider the following perturbation of equation (4.2):

$$y(n+1) = \sum_{s=-\infty}^{n} \{A(n)K(n-s) + B(n)G(s-n)|y(0)|\}y(s) + \sum_{s=-\infty}^{n} H(n,s,y(s)). \tag{4.5}$$

We have the following result.

THEOREM 4.2. *Assume that Conditions (E_1) and (E_3) of Theorem 4.1 held. In addition suppose the following conditions are satisfied.*

(E_4) $H : \mathbb{N}(n_0) \times \mathbb{Z}^- \times \mathbb{C}^e \longrightarrow \mathbb{C}^e$ *is continuous with respect to the third variable.*

(E_5) *There are a positive function* $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *nondecreasing, such that* $\lim_{\delta\to 0^+} \sup_{\gamma\in(0,\delta]} \cdot g(\gamma)/\gamma = 0$ *and a function* $\beta : \mathbb{N}(n_0) \times \mathbb{Z}^- \longrightarrow \mathbb{R}^+$, *such that* $|H(n,\tau,\tilde{\alpha}(\tau)z)| \le a_n\beta(n,\tau)\cdot g(|z|)$, *for all* $n \ge n_0$, $\tau \in \mathbb{Z}^-$, *and* $z \in \mathbb{C}^e$, *where* $a_n \ge 0$, $a_n \longrightarrow 0$, *and* $\sum_{s=n_0}^{\infty}\sum_{\tau=-\infty}^{s} \beta\cdot(s,\tau) < +\infty$.

Then, there are positive constants $\tilde{M}$ *and* $n_1 \in \mathbb{N}(n_0)$, *such that for each* $\varphi \in P(n_1)\mathcal{B}$ *with* $\|\varphi\|_{\mathcal{B}} \le \tilde{M}$, *there is a solution* $y = y(\varphi) = y(n, n_1, \psi)$, *with* $P(n_1)\psi = \varphi$, *of equation (4.5) for* $n \ge n_1$ *such the limit of* $y_\cdot(\varphi)$ *is zero. Moreover, the application* $y_\cdot(\varphi) \longrightarrow Z_\varphi$ *is continuous (see Remark 3.1); the set of all convergent solutions of equation (4.5) is equiconvergent and the set* $\tilde{\Omega}$ *is relatively compact on* X_∞.

We now want to present an example to illustrate the usefulness of Theorem 4.1.

EXAMPLE 4.1. Let $a_i(n)$, $i = 1,2$ be two sequences and σ, γ be a positive constant, such that:

(i) $\rho_1^* := \sup_{n\ge 0}\max_{-n\le\theta\le 0}[[\prod_{s=n+\theta}^{n-1}|a_1(s)|^{-1}]/e^{-\gamma\theta}] < +\infty$.
(ii) $\prod_{s=\tau}^{n}|a_1(s)| \le \sigma$, $0 \le \tau \le n$.
(iii) $\prod_{s=\tau}^{n}|a_2(s)|^{-1} \le \sigma$, $0 \le \tau \le n$.
(iv) $\lim_{n\to\infty}\prod_{s=\tau}^{n}|a_1(s)| = 0$, $\tau \ge 0$.

Some concrete examples of functions a_1 and a_2 satisfying the previous assumptions are:

(a) $a_1(n) := 1/\delta$, $a_2(n) := 2/\delta$ with $1 < \delta \le \min\{2, e^{\gamma}\}$ or $1/\mu \le \delta \le \min\{2\lambda, \nu e^{\gamma}\}$, where $\mu, \lambda, \nu \in (0,1)$.
(b) $\eta < e^{-\gamma} < \mu < 1$, $\gamma > 0$, $a_1(n) := \mu$, $a_2(n) := 1/\eta$.
(c) $1/(\nu e^{\gamma}) \le |a_1(n)| \le \mu$, $1/\lambda \le |a_2(n)|$, for all $n \ge 0$ where $\mu, \lambda, \nu \in (0,1)$.

From now until the end of Example 4.1, we will assume that a_1 and a_2 are functions satisfying (i)–(iv). Using (ii) and (iii), we can assert that

$$\prod_{s=\tau}^{n}|a_2(s)|^{-1} \le \sigma^2\prod_{s=\tau}^{n}|a_1(s)|^{-1}. \tag{4.6}$$

We consider the following nonautonomous difference system:

$$x(n+1) = A(n)x(n), \tag{4.7}$$

where $A(n)$ is a matrix defined by $\mathrm{diag}(a_1(n), a_2(n))$. We begin with a complete analysis to check the dichotomic properties. We recall that $T(n,\tau)$, $n \ge \tau$, is a bounded linear operator on the space $\mathcal{B}_\alpha$, with $\alpha(n) = e^{\gamma n}$, defined by

$$[T(n,\tau)\varphi](\theta) = \begin{cases} \left(\left(\prod_{s=\tau}^{n+\theta-1} a_1(s)\right)\varphi^1(0), \left(\prod_{s=\tau}^{n+\theta-1} a_2(s)\right)\varphi^2(0)\right), & -(n-\tau) \le \theta \le 0, \\ \left(\varphi^1(n-\tau+\theta), \varphi^2(n-\tau+\theta)\right), & \theta < -(n-\tau). \end{cases}$$

A computation shows that

$$T(n,s)T(s,m) = T(n,m), \qquad n \geq s \geq m \quad \text{and} \quad T(n,n) = I.$$

The problem of deciding when a functional difference equation has an ordinary dichotomy is *a priori* much more complicated than in the ordinary difference system case, because it is necessary to construct suitable projections and to get some estimates on the norm of solution operator which acts on the phase space with infinite dimension. In our case the projections can be taken as $P(n) : \mathcal{B}_\alpha \longrightarrow \mathcal{B}_\alpha$ given by

$$[P(n)\varphi](\theta) = \begin{cases} \left(\varphi^1(\theta), \varphi^2(\theta) - \left(\prod_{s=n+\theta}^{n-1} a_2(s)^{-1}\right)\varphi^2(0)\right), & -n \leq \theta \leq 0, \\ \left(\varphi^1(\theta), \varphi^2(\theta)\right), & \theta < -n, \end{cases}$$

and $Q(n) : \mathcal{B}_\alpha \longrightarrow \mathcal{B}_\alpha$ is given by

$$[Q(n)\varphi](\theta) = \begin{cases} \left(0, \left(\prod_{s=n+\theta}^{n-1} a_2(s)^{-1}\right)\varphi^2(0)\right), & -n \leq \theta \leq 0, \\ (0,0), & \theta < -n. \end{cases}$$

For $n \geq \tau$, we observe that $T(n,\tau) : Q(\tau)\mathcal{B}_\alpha \longrightarrow Q(n)\mathcal{B}_\alpha$ is given by

$$[T(n,\tau)Q(\tau)\varphi](\theta) = \begin{cases} \left(0, \left(\prod_{s=\tau}^{n+\theta-1} a_2(s)\right)\varphi^2(0)\right), & -(n-\tau) \leq \theta \leq 0, \\ \left(0, \left(\prod_{s=n+\theta}^{\tau-1} a_2(s)^{-1}\right)\varphi^2(0)\right), & -n \leq \theta \leq -(n-\tau), \\ (0,0), & \theta < -n. \end{cases}$$

We can see that for $n \geq \tau$,

$$\begin{aligned} T(n,\tau)Q(\tau) &= Q(n)T(n,\tau), \\ T(n,\tau)P(\tau) &= P(n)T(n,\tau). \end{aligned}$$

We can prove that $T(n,\tau)$, $n \geq \tau$ is an isomorphism of $Q(\tau)\mathcal{B}_\alpha$ onto $Q(n)\mathcal{B}_\alpha$. We define $T(\tau,n)$ as the inverse mapping, which is given by

$$[T(\tau,n)Q(n)\varphi](\theta) = \begin{cases} \left(0, \left(\prod_{s=\tau+\theta}^{n-1} a_2(s)^{-1}\right)\varphi^2(0)\right), & -\tau \leq \theta \leq 0, \\ (0,0), & \theta < -\tau. \end{cases}$$

By virtue of (4.6), we claim that there is a positive constant $\tilde{K}$, such that

$$\|T(n,\tau)P(\tau)\| \leq \tilde{K}, \qquad n \geq \tau. \tag{4.8}$$

In fact,

$$\|T(n,\tau)P(\tau)\| \leq \max_{-(n-\tau)\leq\theta\leq 0} \left[\frac{\left[\prod_{s=\tau}^{n+\theta-1} |a_1(s)|\right]}{e^{-\gamma\theta}} \right.$$

$$+3 \max_{-n\le\theta\le-(n-\tau)} \left[\frac{\left[\prod_{s=n+\theta}^{\tau-1} |a_2(s)|^{-1}\right]}{e^{-\gamma\theta}} \right]$$

$$\le \left[\prod_{s=\tau}^{n-1} |a_1(s)|\right] \max_{-(n-\tau)\le\theta\le 0} \left[\frac{\left[\prod_{s=n+\theta}^{n-1} |a_1(s)|^{-1}\right]}{e^{-\gamma\theta}} \right]$$

$$+ 3\sigma^2 \left[\prod_{s=\tau}^{n-1} |a_1(s)|\right] \max_{-n\le\theta\le-(n-\tau)} \left[\frac{\left[\prod_{s=n+\theta}^{n-1} |a_1(s)|^{-1}\right]}{e^{-\gamma\theta}} \right]$$

$$\le 6\sigma\rho_1^* \left(\sigma^2+1\right).$$

On the other hand, we can verify that

$$\|T(\tau,n)Q(n)\| \le \rho_2^*\sigma, \tag{4.9}$$

where $\rho_2^* := \sup_{n\ge 0} \max_{-n\le\theta\le 0}[[\prod_{s=n+\theta}^{n-1} |a_2(s)|^{-1}]/e^{-\gamma\theta}]$. From estimates (4.8) and (4.9), we get that the homogeneous system (4.7) admits an ordinary dichotomy, such that $\lim_{n\to\infty} T(n,m)\cdot P(m) = 0$, where $\tilde{K} := 6(\rho_1^*+\rho_2^*)(\sigma^2+1)\sigma$ ($\tilde{K}$ is the constant of Definition 2.1(a)). Next, we consider the following perturbation of system (4.7):

$$y(n+1) = A(n)y(n) + D(n)R(n)|y(0)|y(n) + \nu \sum_{s=-\infty}^{n} D(n)R(s)|y(s)|y(s), \tag{4.10}$$

where $D(n)$ and $R(n)$ are two 2×2 matrices, such that

$$\chi_1 := \sum_{\tau=-\infty}^{-1} |R(\tau)|e^{-2\gamma\tau} < +\infty, \qquad \chi_2 := \sum_{\tau=0}^{\infty} |R(\tau)| < +\infty, \qquad \chi_3 := \sum_{\tau=1}^{\infty} |\tau D(\tau)|e^{\gamma\tau} < +\infty,$$

and ν is a real number small enough. We can verify that if ν satisfies

$$6\left(\rho_1^*+\rho_2^*\right)\left(\sigma^2+1\right)\sigma(\chi_1+\chi_2)\chi_3|\nu| < 1,$$

then Theorem 4.1 is applicable to system (4.10).

EXAMPLE 4.2. We consider the following difference equation:

$$y(n+1) = y(n) + b(n)y(0)y(n) + \nu c(n)|y(n)|y(n), \qquad n\ge n_0\ge 0, \tag{4.11}$$

where $b(n)$ and $c(n)$ are sequences of complex numbers defined for $n\ge n_0$, such that $\rho_b := \sum_{s=n_0}^{\infty} |sb(s)|e^{\gamma s} < +\infty$, $\rho_c := \sum_{s=n_0}^{\infty} |sc(s)|e^{\gamma s} < +\infty$. If we take $P(n) = I$, $\alpha(n) = e^{\gamma n}$, $\gamma > 0$, then (D_6) of Theorem 3.2 holds, with $\tilde{K} = 1$ (see Definition 2.1(a)) and the operator $L(m)$ being defined as $[L(m)\varphi](\theta) = \varphi(0)$, $\theta\in\mathbb{Z}^-$. In fact, we have the following estimate:

$$\|T(n,m) - L(m)\| \le \frac{2}{e^{\gamma(n-m)}}, \qquad n\ge m.$$

Alternatively, we can consider the projections

$$[P(m)\varphi](\theta) = \begin{cases} \varphi(\theta)-\varphi(-m), & -m\le\theta\le 0,\\ \varphi(\theta), & \theta < -m,\end{cases}$$

and

$$[Q(m)\varphi](\theta) = \begin{cases} \varphi(-m), & -m \le \theta \le 0, \\ 0, & \theta < -m. \end{cases}$$

It is easy to verify that $T(n,m)P(m) = P(n)T(n,m)$. Also, we can prove that $T(n,m)$, $n \ge m$ is an isomorphism of $Q(m)\mathcal{B}_\alpha$ onto $Q(n)\mathcal{B}_\alpha$. We define $T(m,n)$ as the inverse mapping, which is given by

$$[T(m,n)Q(n)\varphi](\theta) = \begin{cases} \varphi(-n), & -m \le \theta \le 0, \\ 0, & \theta < -m. \end{cases}$$

Next, we consider $L(m)$ an operator in $\mathcal{L}(\mathcal{B}_\alpha)$ defined by $[L(m)\varphi](\theta) = \varphi(0) - \varphi(-m)$. A straightforward computation show that

$$\|T(n,m)P(m) - L(m)\| \le \frac{3}{e^{\gamma(n-m)}}, \qquad n \ge m.$$

Hence, $\lim_{n\to\infty} T(n,m)P(m) = L(m)$. We can verify that

$$\|T(n,m)P(m)\| \le 2e^{\gamma n}, \qquad n \ge m, \qquad \text{and} \qquad \|T(m,n)Q(n)\| \le 2e^{\gamma n}, \qquad n \ge m.$$

Putting $T_1(n,m) := e^{-\gamma n}T(n,m)$, $n \ge m$. It is easy to see that operator T_1 satisfies (D_1) of Theorem 3.1 with $\tilde{K} = 2$. On the other hand, condition $\rho_b < +\infty$, implies (d-2) (of Assumption (D)) holds. Let ν be a real number, such that $2|\nu|\rho_c < 1$; this condition guarantees (d-4). For equation (4.11) we can get a similar estimate like in equation (4.4), hence (D_2) is satisfied. Therefore by Theorem 3.2 (or Theorem 3.1), there are constants $n_1 \ge n_0$ and R (small enough), such that for each $\varphi \in P(n_1)\mathcal{B}_\alpha[R]$, there is a convergent solution $y = y(\varphi)$ of equation (4.11) for $n \ge n_1$. The set of this kind of solutions whose initial condition is in $P(n_1)\mathcal{B}_\alpha[R]$ is equiconvergent at ∞ on X_∞. Moreover, it is easy to see that (D_4) (see Remark 3.1) and (D_5) (see Remark 3.3) are satisfied, then the correspondence $y.(\varphi) \longrightarrow Z_\varphi$ is bicontinuous, the application $\varphi \longrightarrow y.(\varphi)$ is continuous, and the set $\tilde{\Omega}$ is relatively compact.

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